

ISOMETRIC IMMERSION OF COMPLETE SURFACES WITH SLOWLY DECAYING NEGATIVE GAUSS CURVATURE

WENTAO CAO, FEIMIN HUANG, AND DEHUA WANG

ABSTRACT. The isometric immersion of two-dimensional Riemannian manifolds or surfaces in the three-dimensional Euclidean space is a fundamental problem in differential geometry. When the Gauss curvature is negative, the isometric immersion problem is considered in this paper through the Gauss-Codazzi system for the second fundamental forms. It is shown that if the Gauss curvature satisfies an integrability condition, the surface has a global smooth isometric immersion in the three-dimensional Euclidean space even if the Gauss curvature decays very slowly at infinity. The new idea of the proof is based on the novel observations on the decay properties of the Riemann invariants of the Gauss-Codazzi system. The weighted Riemann invariants are introduced and a comparison principle is applied with properly chosen control functions.

1. INTRODUCTION

The isometric embedding or immersion of Riemannian manifolds is a fundamental problem in differential geometry and has been studied extensively since the 19th century. Roughly speaking, given a Riemannian manifold (M^n, \mathbf{g}) , the isometric embedding or immersion problem is to seek a mapping \mathbf{r} into \mathbb{R}^m such that

$$d\mathbf{r} \cdot d\mathbf{r} = \mathbf{g}. \quad (1.1)$$

Since the number of equations in (1.1) is $\frac{n(n+1)}{2}$, it is believed that the critical dimension for isometric embedding or immersion is $m = s_n = \frac{n(n+1)}{2}$ (called the Janet dimension). There have been a number of remarkable achievements in the case $m > s_n$; see Nash [32, 33], Gromov [15, 16] and the references therein. It is a classical and challenging problem to study the critical dimensional case, that is, $m = s_n$. In this direction, for the analytic metric \mathbf{g} , the local embedding problem was solved by Janet [26] in 1926 and Cartan [5] in 1927. For the smooth metric \mathbf{g} , the local smooth isometric immersion was studied in Bryant-Griffiths-Yang [1], Goodman-Yang [14], Nakamura-Maeda [30, 31], Poole [36], and Chen-Clelland-Slemrod-Wang-Yang [6] when $n = 3$ and in [1, 14] when $n = 4$. For more discussions see [6] for $n \geq 3$ and [19] for $n = 2$.

Date: June 1, 2016.

2010 Mathematics Subject Classification. 53C42, 53C21, 53C45, 35L45.

Key words and phrases. Isometric immersion of surfaces, the Gauss-Codazzi system, second fundamental forms, smooth solutions, negative Gauss curvature, slowly decay, Riemann invariants.

In this paper, we focus on the two-dimensional problem, i.e. $n = 2, s_n = 3$. In this extremely interesting case, the problem can be reduced to solve the Darboux equation/Gauss-Codazzi system (c.f. [2, 19, 37]). It is well known that the Darboux equation/Gauss-Codazzi system is of mixed type of partial differential equations depending on the sign of the Gauss curvature K ; that is, the system is elliptic if $K > 0$, hyperbolic if $K < 0$, and is of mixed type if K changes sign. There have been considerable progresses on the local smooth embedding [18, 20, 21, 27, 28] and the global smooth embedding [17, 22, 25, 34, 39] (for the case $K \geq 0$). Dong [11] also studied a semi-global isometric immersion. We refer the reader to the book of Han-Hong [19] for an excellent review and discussions of the smooth isometric embedding or immersion of two-dimensional manifolds (surfaces) into the three-dimensional Euclidean space. We remark that there are some recent works on the $C^{1,1}$ isometric immersions via compensated compactness in [3, 4, 7–9].

The present paper is concerned with the smooth isometric embedding/immersion of surfaces with negative Gauss curvatures. When the Gauss curvature is negative, i.e., $K < 0$, the global embedding problem was first investigated by Hilbert [23], where he gave a negative answer for the hyperbolic plane whose Gauss curvature is a negative constant. Efimov showed in [12] that there is no C^2 isometric immersion in \mathbb{R}^3 if the Gauss curvature is bounded above by a negative constant, and then showed further in [13] that if the Gauss curvature satisfies

$$\left| \nabla \frac{1}{k} \right| \leq q_1, \text{ for some constant } q_1 > 0,$$

where $K = -k^2$, there is no C^3 isometric immersion in \mathbb{R}^3 . It is very challenging to prove the existence of global smooth isometric immersion of a two-dimensional Riemannian manifold with negative Gauss curvature in \mathbb{R}^3 . Yau [40] proposed the following problem: “Find a sufficient condition for a complete negative curved surface to be isometrically embedded in \mathbb{R}^3 ”. Hong first gave a positive answer in his groundbreaking work [24] as follows.

Theorem (Hong’s Theorem in [24], 1993). *Let $(\mathcal{M}, \mathbf{g})$ be a smooth complete simply connected 2-dimensional surface with negative Gauss curvature $K = -k^2(\theta, \rho)$. Assume that k satisfies*

$$\partial_\rho \ln(k\rho^{1+\delta}) \leq 0 \text{ as } \rho \geq R, \quad (1.2)$$

for some positive constants δ and R , and

$$\partial_\theta^i \ln k \ (i = 1, 2), \ \rho \partial_\theta \partial_\rho \ln k \text{ bounded.}$$

Then $(\mathcal{M}, \mathbf{g})$ has a smooth isometric immersion in \mathbb{R}^3 .

Remark 1.1. In the above theorem, (θ, ρ) denotes the geodesic polar coordinates. From the condition (1.2), the decay rate of the Gauss curvature k at infinity is

$$k \approx \frac{1}{\rho^{1+\delta}}, \text{ for some } \delta > 0. \quad (1.3)$$

On the other hand, Efimov [13] showed that there is no C^3 isometric immersion if $k = \frac{1}{\rho}$. Thus Hong's decay rate (1.2) is almost optimal!

Although Hong's decay rate is almost optimal, there is still a gap between $\frac{1}{\rho}$ and $\frac{1}{\rho^{1+\delta}}$. For instance, the case that

$$k = \frac{1}{\rho(\ln \rho)^2} \quad \text{or equivalently} \quad K = -\frac{1}{\rho^2(\ln \rho)^4} \quad \text{for } \rho \text{ large,} \quad (1.4)$$

is excluded in Hong [24]. How to relax the decay rate (1.2) has become a longstanding open problem since 1993. In fact, Hong in [24] raised himself the following question: “One wonders if the restriction on the rate of the decay of the curvature as fast as the geodesic distance from the base curve of $-(2+\delta)$ order at infinity, can be relaxed. It is still open”. The purpose of this paper is to relax Hong's decay rate (1.2) for global smooth isometric immersion. Precisely, our main result can be stated as follows.

Theorem 1.1 (Main Theorem). *Let $(\mathcal{M}, \mathbf{g})$ be a smooth complete simply connected 2-dimensional surface with the metric $\mathbf{g} = G^2(\theta, \rho)d\theta^2 + d\rho^2$ in the geodesic polar coordinate (θ, ρ) , the Gauss curvature of which $K = -k^2(\theta, \rho)$ satisfies the following conditions:*

- (A1) $\sup_{\theta} \int_{\rho \geq R} k(\theta, \rho) d\rho < \infty$, $k = o(\frac{1}{\rho})$, and $\partial_{\rho} \ln(k\rho^{1+\delta}) \geq 0$, when $\rho \geq R$, for some positive constants R and $\delta \in (0, 1/2)$;
- (A2) $\partial_{\theta}^i \ln k$, $i = 1, 2$, $\rho \partial_{\theta} \partial_{\rho} \ln k$ are bounded,

Then $(\mathcal{M}, \mathbf{g})$ has a smooth isometric immersion in \mathbb{R}^3 .

Remark 1.2. The notation $o(\frac{1}{\rho})$ in the above theorem means $\rho o(\frac{1}{\rho}) \rightarrow 0$ as $\rho \rightarrow \infty$.

Remark 1.3. From (A1), the case (1.4) is included in Theorem 1.1. Thus we give a positive answer to the question raised by Hong [24].

Remark 1.4. Since the condition $\partial_{\rho} \ln(k\rho^{1+\delta}) < 0$ was treated in [24], we consider the other side $\partial_{\rho} \ln(k\rho^{1+\delta}) \geq 0$ where the Gauss curvature may decay slowly.

Remark 1.5. As suggested by Yau [40], the decay rate of curvature should be replaced by an integrability condition. The integrability condition $\sup_{\theta} \int_{\rho \geq R} k(\theta, \rho) d\rho < \infty$ of (A1) is derived from an observation of the special solutions to the Gauss-Codazzi system, see Section 2 below. This integrability condition might be optimal in terms of the decay rate of Gauss curvature at infinity.

As remarked above, the main contribution of this paper is that the decay rate of Hong [24] is relaxed so that a curvature like (1.4) between the gap of $\frac{1}{\rho}$ and $\frac{1}{\rho^{1+\delta}}$ is included in our existence theorem of global smooth isometric immersion of surfaces in \mathbb{R}^3 . To prove the result in Theorem 1.1, new ideas are developed based on our novel observations and insights into the behavior of the Riemann invariants of the Gauss-Codazzi system. We now explain the main strategy. First, we consider a special case that the metric only depends on one variable t , thus the Gauss-Codazzi

system is reduced to an ODE system for the Riemann invariants (w, z) . Then we obtain an explicit formula of solution to the ODE system. From the explicit formula, we have two new important observations: (a) when the curvature k decays slowly than t^{-2} , both w and z decays at the same order as k , i.e., $w \approx k, z \approx k$ (see Section 2 below); (b) $w + z \approx t^{-2}$ decays faster than $w - z \approx k$, which means that some cancellations happen in $w + z$. We expect that the above two properties still hold for general cases. The two observations are the crucial motivations for our approach of proving Theorem 1.1. First novel ingredient of our proof is that we introduce two weighted Riemann invariants $r = \frac{w}{k}, s = \frac{z}{k}$ and derive an equivalent system (3.6) for (r, s) . Based on the comparison principle for linear hyperbolic system, we can show that the first property (a) holds under some additional *a priori* assumption of derivatives $\tilde{r} = (r - s)r_x, \tilde{s} = (r - s)s_x$. The *a priori* assumption on the derivatives can then be verified by the fact that $r + s$ decays faster than $r - s$ at infinity, which is another novel part of our proof. Using the approach above we can finally show that there is a global smooth isometric immersion of surfaces with slowly decaying Gauss curvature, which relaxes the decay rate of curvatures in Hong [24].

The rest of the paper is organized as follows. In Section 2 we provide some basic formulas related to the Gauss-Codazzi system for surfaces with negative Gauss curvature. Then we give an explicit formula of special solutions to the Gauss-Codazzi system. The explicit formula of the special solutions provides us important observations that motivate our new ideas. Section 3 is devoted to a key theorem for the existence of global smooth isometric immersion of surfaces in the geodesic coordinates. This section contains our main new ideas and show why we can relax the decay rate. The main Theorem 1.1 in geodesic polar coordinates is proved in Section 4 following the original idea of Hong [24] with modifications through variable transformations.

2. SPECIAL SOLUTION TO THE GAUSS-CODAZZI SYSTEM

The Gauss-Codazzi system for the isometric immersion or embedding of surfaces into \mathbb{R}^3 is (cf. [19, 24]):

$$\begin{aligned} \partial_{x_2} L - \partial_{x_1} M &= \Gamma_{12}^1 L + (\Gamma_{12}^2 - \Gamma_{11}^1) M - \Gamma_{11}^2 N, \\ \partial_{x_2} M - \partial_{x_1} N &= \Gamma_{22}^1 L + (\Gamma_{22}^2 - \Gamma_{21}^1) M - \Gamma_{21}^2 N, \\ LN - M^2 &= K|\mathbf{g}|, \end{aligned} \tag{2.1}$$

where L, M and N denote the coefficients of the second fundamental form

$$\mathcal{I} = Ldx_1^2 + 2Mdx_1dx_2 + Ndx_2^2, \tag{2.2}$$

with the given metric $\mathbf{g} = g_{ij}dx_i dx_j, i, j = 1, 2$, of the surface; $\Gamma_{jk}^i, i, j, k = 1, 2$, are the Christoffel symbols, $|\mathbf{g}|$ is the determinant of the metric matrix (g_{ij}) , and K is the Gauss curvature.

In this paper, we consider the isometric immersion of a complete simply connected two-dimensional manifold (or a surface) with the negative Gauss curvature

$$K = -k^2$$

for some smooth function $k > 0$. Then the Gauss-Codazzi system (2.1) is hyperbolic, the two eigenvalues are

$$\lambda_+ = \frac{-M + k\sqrt{|\mathbf{g}|}}{L}, \quad \lambda_- = \frac{-M - k\sqrt{|\mathbf{g}|}}{L},$$

and the corresponding Riemann invariants (cf. [38]) are

$$w = \frac{-M + k\sqrt{|\mathbf{g}|}}{L}, \quad z = \frac{-M - k\sqrt{|\mathbf{g}|}}{L}.$$

For simplicity of notations, let $x = x_1$, $t = x_2$. A direct calculation leads to the following system for the Riemann invariants w and z (cf. [19]):

$$\begin{aligned} w_t + zw_x &= \frac{w - z}{2}(\partial_t + w\partial_x) \ln k - \Gamma_{22}^1 + (\Gamma_{22}^2 - \Gamma_{12}^1)w \\ &\quad - \Gamma_{12}^1 z + \Gamma_{12}^2 w^2 + (\Gamma_{12}^2 - \Gamma_{11}^1)wz + \Gamma_{11}^2 w^2 z, \\ z_t + wz_x &= \frac{z - w}{2}(\partial_t + z\partial_x) \ln k - \Gamma_{22}^1 + (\Gamma_{22}^2 - \Gamma_{12}^1)z \\ &\quad - \Gamma_{12}^1 w + \Gamma_{12}^2 z^2 + (\Gamma_{12}^2 - \Gamma_{11}^1)wz + \Gamma_{11}^2 wz^2. \end{aligned} \tag{2.3}$$

The system (2.3) is equivalent to (2.1) when $w > z$. Note that (2.3) is linearly degenerate. We shall show that the system (2.3) admits a global smooth solution under certain conditions.

In particular, in this section we explore some special solution to the Gauss-Codazzi system, which provides us interesting observations and motivations on the conditions for the global existence of smooth solutions. Consider a special case that the metric is of the form $\mathbf{g} = B^2(t)dx^2 + dt^2$, then the corresponding Christoffel symbols are

$$\begin{aligned} \Gamma_{12}^1 &= \frac{B'}{B}, \quad \Gamma_{11}^2 = -BB', \\ \Gamma_{11}^1 &= \Gamma_{22}^1 = \Gamma_{12}^2 = \Gamma_{22}^2 = 0, \end{aligned} \tag{2.4}$$

where $'$ denotes $\frac{d}{dt}$. When the initial data of the system (2.3) is given by $w(0) = w_0$, $z(0) = z_0$ with w_0 and z_0 both constant, we consider the special solution $(w, z)(t)$ to (2.3). Plugging (2.4) into (2.3), we obtain the following ODE system:

$$\begin{aligned} \frac{dw}{dt} &= -\left(\frac{B'}{B} - \frac{k'}{2k}\right)w - \left(\frac{B'}{B} + \frac{k'}{2k}\right)z - BB'w^2z, \\ \frac{dz}{dt} &= -\left(\frac{B'}{B} - \frac{k'}{2k}\right)z - \left(\frac{B'}{B} + \frac{k'}{2k}\right)w - BB'wz^2. \end{aligned} \tag{2.5}$$

We now solve the ODE system (2.5) that has cubic nonlinear source terms. Note that adding and subtracting the two equations of (2.5) yield

$$\begin{aligned}\frac{d(w+z)}{dt} &= -\frac{2B'}{B}(w+z) - BB'wz(w+z), \\ \frac{d(w-z)}{dt} &= \frac{k'}{k}(w-z) - BB'wz(w-z).\end{aligned}\tag{2.6}$$

Then we get

$$\begin{aligned}w &= \left(\frac{w_0 + z_0}{2B^2} + \frac{k(w_0 - z_0)}{2k(0)} \right) \exp \left[\int_0^t -BB'wz ds \right], \\ z &= \left(\frac{w_0 + z_0}{2B^2} - \frac{k(w_0 - z_0)}{2k(0)} \right) \exp \left[\int_0^t -BB'wz ds \right].\end{aligned}$$

It holds that

$$wz = \left(\frac{(w_0 + z_0)^2}{4B^4} - \frac{k^2(w_0 - z_0)^2}{4k^2(0)} \right) \exp \left[\int_0^t -2BB'wz ds \right].$$

Let $X := wz$. We have

$$\frac{d \ln X}{dt} = \frac{d \ln a(t)}{dt} - 2BB'X, \quad a(t) = \frac{(w_0 + z_0)^2}{4B^4} - \frac{k^2(w_0 - z_0)^2}{4k^2(0)},$$

then

$$X(t) = \frac{\frac{(w_0 + z_0)^2}{4B^4} - \frac{k^2(w_0 - z_0)^2}{4k^2(0)}}{1 + \frac{1}{4}(w_0 + z_0)^2(1 - \frac{1}{B^2}) - \frac{1}{4}(w_0 - z_0)^2B'^2},$$

where we have used the Gauss equation $B'' = k^2B$. Thus we obtain a special solution to the Gauss-Codazzi system (2.3) of the following form:

$$w = \frac{k(0)(w_0 + z_0) + B^2k(w_0 - z_0)}{\sqrt{4B^4k(0)^2 + k^2(0)(w_0 + z_0)^2(B^4 - B^2) - k^2(0)(w_0 - z_0)^2B^4B'^2}}, \tag{2.7}$$

$$z = \frac{k(0)(w_0 + z_0) - B^2k(w_0 - z_0)}{\sqrt{4B^4k(0)^2 + k^2(0)(w_0 + z_0)^2(B^4 - B^2) - k^2(0)(w_0 - z_0)^2B^4B'^2}}. \tag{2.8}$$

When k decays at a slower rate than t^{-2} for large t , we have the following two important observations:

- (1) both w and z decays at the same order as k , i.e., $w \approx k, z \approx k$;
- (2) $w + z \approx t^{-2}$ decays faster than $w - z \approx k$.

On the other hand, from (2.7) and (2.8), the solution may blow up in a finite time if B' is not bounded. Note that B' has the following estimate as in [24]:

$$\int_0^t k^2 dt \leq B'(t) \leq \int_0^t k^2 ds \exp \left[\int_0^t s k^2 ds \right].$$

Therefore the ODE system (2.5) has a global solution only if

$$\int_0^\infty k^2(t)tdt < +\infty. \quad (2.9)$$

It seems that Hong's decay rate (1.3) could be relaxed by (2.9).

However, the metric generally depends not only on t but also on x . In general, the Christoffel symbols are

$$\begin{aligned} \Gamma_{11}^1 &= \frac{B_x}{B}, & \Gamma_{11}^2 &= -BB_t, & \Gamma_{22}^1 &= 0, \\ \Gamma_{12}^1 &= \frac{B_t}{B}, & \Gamma_{12}^2 &= \Gamma_{22}^2 = 0, \end{aligned} \quad (2.10)$$

then the Gauss-Codazzi system (2.3) contains some quadratic terms w^2, wz, z^2 . Note that $\partial_x \ln B = \frac{B_x}{B}$ and $\partial_x \ln k = \frac{k_x}{k}$ are uniformly bounded (see Lemma 3.1 below), all the coefficients of quadratic terms in the Gauss-Codazzi system (2.3) are uniformly bounded. We consider the following toy model of (2.3), i.e. the ODE system (2.5) with additional quadratic terms:

$$\begin{aligned} \frac{dw}{dt} &= -\left(\frac{B_t}{B} - \frac{k_t}{2k}\right)w - \left(\frac{B_t}{B} + \frac{k_t}{2k}\right)z - BB_tw^2z + w^2, \\ \frac{dz}{dt} &= -\left(\frac{B_t}{B} - \frac{k_t}{2k}\right)z - \left(\frac{B_t}{B} + \frac{k_t}{2k}\right)w - BB_twz^2 - z^2, \end{aligned} \quad (2.11)$$

which can be reduced to a scalar equation

$$\frac{dw}{dt} = \frac{k_t}{k}w - BB_tw^2z + w^2, \quad (2.12)$$

for the special case $w = -z$. It is not difficult to show that the equation (2.12) has a global solution only if $\int_0^\infty k(t)dt < \infty$. So we instead use

$$\sup_x \int_0^\infty k(t)dt < +\infty \quad (2.13)$$

to relax Hong's decay rate (1.3). Such decay rate in (2.13) might be optimal for admitting a global immersion in \mathbb{R}^3 , based on the above observation of special solution to the Gauss-Codazzi system (2.1).

3. ISOMETRIC IMMERSION IN GEODESIC COORDINATES

In this section, we prove the isometric immersion in geodesic coordinates under the similar conditions of Theorem 1.1 based on the two important observations in the previous Section 2. This section contains our main new ideas and contributions and show why we can relax the decay rate in Hong [24] of Gauss curvature of surfaces for the global existence of smooth isometric immersion.

From Theorem B in [24], the geodesic coordinate system covers the whole complete simply connected two-dimensional surface with a negative Gauss curvature. The metric of the two-dimensional surface in the geodesic coordinates is of the form

$$\mathbf{g} = B^2(x, t)dx^2 + dt^2,$$

where $B(x, t) > 0$ satisfies

$$\begin{cases} B_{tt} = k^2 B, \\ B(x, 0) = 1, \quad B_t(x, 0) = 0. \end{cases} \quad (3.1)$$

We have the following theorem on the global isometric immersion in the geodesic coordinates.

Theorem 3.1. *Let $(\mathcal{M}, \mathbf{g})$ be a complete simply connected smooth two-dimensional surface with the metric $\mathbf{g} = B^2(x, t)dx^2 + dt^2$ and the Gauss curvature $K = -k^2(x, t)$ satisfying*

- (H1) $\sup_x \int_{|t| \geq T} k(x, t)dt < \infty$, $k = o\left(\frac{1}{|t|}\right)$, $t\partial_t \ln k(|t|^{1+\delta}) \geq 0$ when $|t| \geq T$ for some positive constant T and $\delta \in (0, 1/2)$;
- (H2) $\partial_x^i \ln k$, $i = 1, 2$, $t\partial_x \partial_t \ln k$ are bounded;
- (H3) $\inf_x \int_0^\infty k^2(x, t)dt$ and $\inf_x \int_{-\infty}^0 k^2(x, t)dt$ are positive.

Then $(\mathcal{M}, \mathbf{g})$ admits a smooth isometric immersion in \mathbb{R}^3 .

Remark 3.1. The condition (H1) is motivated by the observations in Section 2. Comparing with (H_1) of Theorem C in [24]:

- (H_1) $k > 0$ and $t\partial_t \ln k(|t|^{1+\delta}) \leq 0$, when $|t| \geq T$, for some large enough positive constant T and $\delta \in (0, 1)$,

the condition (H1) in Theorem 3.1 allows slower decay rate for the Gauss curvature. For instance, let

$$k = \frac{1}{(t+2)(\ln(t+2))^2}, \quad t \gg 1, \quad (3.2)$$

it is straightforward to check that (3.2) is included in (H1) of Theorem 3.1, while excluded in (H_1) of Theorem C in [24]. Thus Theorem 3.1 gives a positive answer to the question raised in [24]: “One wonders if the restriction on the rate of the decay of the curvature as fast as the geodesic distance from the base curve of $-(2+\delta)$ order at infinity, can be relaxed.”

We first derive some properties of $B(x, t)$ under the conditions of Theorem 3.1.

Lemma 3.1. *Under the conditions of Theorem 3.1, one has the following estimate:*

$$\begin{aligned} \frac{B_t}{B} &= \frac{1}{t} + o\left(\frac{1}{|t|}\right) \text{ uniformly in } x \text{ for } |t| \text{ sufficiently large, and} \\ \partial_x^i \ln B, i &= 1, 2, \quad B\partial_x \partial_t \ln B \text{ are bounded for } |t| \text{ sufficiently large.} \end{aligned}$$

Proof. We only consider the case $t \geq 0$ since $t \leq 0$ can be treated similarly. From Lemma 1.1 in [24], we have for fixed x ,

$$\int_0^t k^2(x, t) dt \leq B_t \leq \int_0^t k^2(x, s) ds \exp \left[\int_0^t s k^2(x, s) ds \right].$$

On the other hand, by (H1) and (H3), one has

$$\begin{aligned} \sup_x \int_0^t k^2(x, s) s ds &< C \sup_x \int_0^t k(x, s) ds < \infty, \\ \inf_x \int_0^t k^2(x, t) dt &\leq B_t \leq \sup_x \int_0^t k^2(x, s) ds \exp \left[\sup_x \int_0^t s k^2(x, s) ds \right], \end{aligned}$$

then there exist positive constant C_1 and C_2 such that

$$C_1 |t| \leq B(x, t) \leq C_2 |t|$$

for large $|t|$. In addition, from (3.1) we have

$$\partial_t \partial_t \ln B = k^2 - (\partial_t \ln B)^2, \quad (3.3)$$

which implies that

$$|\partial_t \ln B| \leq \int_0^t k^2(x, s) ds \leq C.$$

Differentiating (3.3) with respect to x yields the following equation for $\partial_x \partial_t \ln B(x, t)$,

$$\partial_t (\partial_x \partial_t \ln B) = 2k k_x - 2\partial_x \partial_t \ln B \partial_t \ln B,$$

then

$$\partial_x \partial_t \ln B = \frac{1}{B^2} \int_0^t 2k k_x B^2 ds.$$

Noting that $B_x(x, 0) = 0$, we have

$$\begin{aligned} |\partial_x \ln B| &= \left| \int_0^t \frac{1}{B^2} \int_0^s 2k k_x B^2 d\tau ds \right| \leq C \int_0^t \frac{1}{B^2(x, s)} \int_0^s k^2(x, \tau) B^2(x, \tau) d\tau ds \\ &= C \int_0^t k^2(x, \tau) B^2(x, \tau) \int_\tau^t \frac{1}{B^2(x, s)} ds d\tau \leq C \int_0^t k^2(x, \tau) \tau d\tau \leq C, \end{aligned}$$

where we have used the condition (H2), that is, $|k_x| \leq Ck$. Hereafter C denotes a generic positive constant. Similarly, we have

$$\begin{aligned} |\partial_x \partial_x \ln B| &= \left| \partial_x \left(\int_0^t \frac{1}{B^2} \int_0^s 2k k_x B^2 d\tau ds \right) \right| \leq \left| \int_0^t \frac{2B_x}{B^3} \int_0^s k k_x B^2 d\tau ds \right| \\ &\quad + \left| \int_0^t \frac{1}{B^2} \int_0^s (2k_x k_x B^2 + 2k k_{xx} B^2 + 4k k_x B B_x) d\tau ds \right| \\ &\leq C \int_0^t \frac{1}{B^2} \int_0^s k^2 B^2 d\tau ds \leq C, \end{aligned}$$

and

$$|B\partial_x\partial_t\ln B| \leq \frac{1}{B} \int_0^t 2k|k_x|B^2ds \leq C \int_0^t sk^2(x,s)ds \leq C.$$

Thus, $\partial_x^i \ln B, i = 1, 2$ and $B\partial_x\partial_t \ln B$ are bounded for large $|t|$.

Next we shall study the asymptotic behavior of $\partial_t \ln B$ for large t . Since $B_{tt} > 0$ and B_t is bounded, $B_t(x, \infty)$ exists. Moreover,

$$\begin{aligned} B_t(x, t) &= B_t(x, \infty) - \int_t^\infty Bk^2(x, s)ds, \\ B(x, t) &= 1 + B_t(x, \infty)t - t \int_t^\infty k^2Bds - \int_0^t k^2Bsd s. \end{aligned}$$

Then

$$\begin{aligned} \frac{B_t}{B} &= \frac{B_t(x, \infty) - \int_t^\infty Bk^2(x, s)ds}{1 + B_t(x, \infty)t - t \int_t^\infty k^2Bds - \int_0^t Bk^2sd s} \\ &= \frac{1}{t} + \frac{1}{t} \frac{t^{-1} \int_0^t Bk^2sd s - t^{-1}}{t^{-1} + B_t(x, \infty) - \int_t^\infty k^2Bds - t^{-1} \int_0^t Bk^2sd s}. \end{aligned}$$

From condition (H1), $kt^{1+\delta}$ is increasing for large t with $\delta \in (0, \frac{1}{2})$. Then it follows from $k = o(\frac{1}{|t|})$ that

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^t k^2s^2ds \leq \lim_{t \rightarrow \infty} k^2t^{1+2\delta} \int_0^t s^{-2\delta}ds \leq \lim_{t \rightarrow \infty} \frac{1}{1-2\delta} k^2t^2 = 0. \quad (3.4)$$

Noting that

$$B_t(x, \infty) \geq \inf_x \int_0^\infty k^2(x, s)ds > 0,$$

one has

$$\lim_{t \rightarrow \infty} \frac{t^{-1} \int_0^t Bk^2sd s - t^{-1}}{t^{-1} + B_t(x, \infty) - \int_t^\infty k^2Bds - t^{-1} \int_0^t Bk^2sd s} = 0,$$

that is, $\frac{B_t}{B} = \frac{1}{t} + o\left(\frac{1}{|t|}\right)$. This completes the proof of Lemma 3.1. \square

Now we turn to the proof of Theorem 3.1. Plugging all the Christoffel symbol formulas (2.10) into (2.3), we obtain

$$\begin{aligned} w_t + zw_x &= -\left(\frac{B_t}{B} - \frac{k_t}{2k}\right)w - \left(\frac{B_t}{B} + \frac{k_t}{2k}\right)z - BB_tw^2z - \left(\frac{B_x}{B} + \frac{k_x}{2k}\right)wz + \frac{k_x}{2k}w^2, \\ z_t + wz_x &= -\left(\frac{B_t}{B} - \frac{k_t}{2k}\right)z - \left(\frac{B_t}{B} + \frac{k_t}{2k}\right)w - BB_twz^2 - \left(\frac{B_x}{B} + \frac{k_x}{2k}\right)wz + \frac{k_x}{2k}z^2. \end{aligned} \quad (3.5)$$

Motivated by the observation in Section 2 that w, z decay at the same order as k for the ODE system (2.5), we make the following variable transformation to introduce the weighted Riemann invariants:

$$r = \frac{w}{k} \quad \text{and} \quad s = \frac{z}{k},$$

then (3.5) is rewritten as

$$\begin{aligned} r_t + ksr_x &= - \left(\frac{B_t}{B} + \frac{k_t}{2k} \right) (r + s) - \left(\frac{B_x}{B} + \frac{3k_x}{2k} \right) krs + \frac{k_x}{2k} kr^2 - BB_t k^2 r^2 s \\ &=: f(r, s, x, t), \\ s_t + krs_x &= - \left(\frac{B_t}{B} + \frac{k_t}{2k} \right) (r + s) - \left(\frac{B_x}{B} + \frac{3k_x}{2k} \right) krs + \frac{k_x}{2k} ks^2 - BB_t k^2 rs^2 \\ &=: g(r, s, x, t). \end{aligned} \tag{3.6}$$

The system (2.3) is equivalent to (3.6) if $r > s$ holds. We focus on the system (3.6) from now on. Denote

$$\partial_\beta = \partial_t + ks\partial_x, \quad \partial_\alpha = \partial_t + kr\partial_x,$$

then we have the following system along the characteristics:

$$\partial_\beta r = f, \quad \partial_\alpha s = g. \tag{3.7}$$

We now derive a system for $r - s$ similar to the ODE system (2.11). Subtracting the second equation from the first one in the system (3.6) yields

$$\begin{aligned} \partial_\beta(r - s) &= (r - s)Q + k\tilde{s}, \\ \partial_\alpha(r - s) &= (r - s)Q + k\tilde{r}, \end{aligned} \tag{3.8}$$

where

$$\tilde{r} =: (r - s)r_x, \quad \tilde{s} =: (r - s)s_x, \tag{3.9}$$

and

$$Q = \frac{f - g}{r - s} = -BB_t k^2 rs + \frac{k_x}{2k}(r + s)k. \tag{3.10}$$

Since the system (3.7) has two characteristics, we have additional derivative terms \tilde{r}, \tilde{s} in (3.8). Similarly, we have the following system for $r + s$:

$$\begin{aligned} \partial_\beta(r + s) &= -k\tilde{s} - (r + s) \left[\left(\frac{2B_t}{B} + \frac{k_t}{k} \right) + BB_t k^2 rs \right] \\ &\quad - 2 \left(\frac{B_x}{B} + \frac{3k_x}{2k} \right) krs + \frac{k_x}{2k} k(r^2 + s^2), \\ \partial_\alpha(r + s) &= k\tilde{r} - (r + s) \left[\left(\frac{2B_t}{B} + \frac{k_t}{k} \right) + BB_t k^2 rs \right] \\ &\quad - 2 \left(\frac{B_x}{B} + \frac{3k_x}{2k} \right) krs + \frac{k_x}{2k} k(r^2 + s^2). \end{aligned} \tag{3.11}$$

Differentiating (3.6) with respect to x , we derive the following system for \tilde{r} and \tilde{s} :

$$\begin{aligned}\partial_\beta \tilde{r} &= (Q + f_r - k_x s) \tilde{r} + f_s \tilde{s} + (r - s) \delta_x f, \\ \partial_\alpha \tilde{s} &= g_r \tilde{r} + (Q + g_s - k_x r) \tilde{s} + (r - s) \delta_x g,\end{aligned}\tag{3.12}$$

where

$$\begin{aligned}f_r(r, s, x, t) &= g_s(s, r, x, t) = -\frac{k_t}{2k} - \frac{B_t}{B} - 2BB_t k^2 r s - \left(\frac{B_x}{B} + \frac{3k_x}{2k}\right) k s + \frac{k_x}{k} k r, \\ f_s(r, s, x, t) &= g_r(s, r, x, t) = -\frac{k_t}{2k} - \frac{B_t}{B} - BB_t k^2 r^2 - \left(\frac{B_x}{B} + \frac{3k_x}{2k}\right) k r,\end{aligned}\tag{3.13}$$

and $\delta_x f$ denotes the derivative with respect to x of the coefficients of r and s in f , i.e.

$$\begin{aligned}\delta_x f &= -\left(\frac{B_t}{B} + \frac{k_t}{2k}\right)_x (r + s) - \left[\left(\frac{B_x}{B} + \frac{3k_x}{2k}\right) k\right]_x r s + \left(\frac{k_x}{2k}\right)_x r^2 - (BB_t k^2)_x r^2 s, \\ \delta_x g &= -\left(\frac{B_t}{B} + \frac{k_t}{2k}\right)_x (r + s) - \left[\left(\frac{B_x}{B} + \frac{3k_x}{2k}\right) k\right]_x r s + \left(\frac{k_x}{2k}\right)_x s^2 - (BB_t k^2)_x r s^2.\end{aligned}\tag{3.14}$$

Now we are ready to prove Theorem 3.1. The strategy is based on the local existence and *a priori* estimates. The local existence can be obtained by a lemma of [24] that holds for general hyperbolic systems. To derive *a priori* estimates, we consider the following hyperbolic system:

$$\begin{aligned}u_t + \lambda_1 u_x &= a_{11} u + a_{12} v + R_1, \\ v_t + \lambda_2 v_x &= a_{21} u + a_{22} v + R_2,\end{aligned}\tag{3.15}$$

with the initial data:

$$u(x, 0) = \varphi_1(x), \quad v(x, 0) = \varphi_2(x),$$

where λ_i , a_{ij} , R_i , $i, j = 1, 2$, are all C^1 smooth functions of (x, t) , and $\lambda_1 \leq \lambda_2$. For each λ_i , from any point (x, t) , we can draw a backward characteristic curve $X = \Gamma_i(\tau; x, t)$ defined by the following ODE:

$$\frac{dX}{d\tau} = \lambda_i(X, \tau), \quad \tau \leq t, \quad \text{with } X(t) = x.\tag{3.16}$$

It is obvious that $\Gamma_2(\tau; x, t) \leq \Gamma_1(\tau; x, t)$, for $0 < \tau < t < T$.

For each point $P = (x^*, t^*)$, denoted by $P(x^*, t^*)$, $t^* \in (0, T]$, we can draw a backward characteristic triangle Δ_P^0 ,

$$\Delta_P^0 = \{(x, t) : \Gamma_2(t; x^*, t^*) \leq x \leq \Gamma_1(t; x^*, t^*), 0 \leq t \leq t^*\}.$$

Denote

$$H = \max_{(x, t) \in \Delta_P^0} \{|a_{ij}|, |\partial_x \lambda_i|, |\partial_x a_{ij}|, i, j = 1, 2\},$$

and $I(\tau) = [\Gamma_2(\tau; x^*, t^*), \Gamma_1(\tau; x^*, t^*)]$. Then we have the following lemma from [24]:

Lemma 3.2 (Hong [24]). *Let (u, v) be a C^1 smooth solution to (3.15) with $R_i = 0, i = 1, 2$ in Δ_P^0 . Then for $(x, t) \in \Delta_P^0$,*

$$|u|, |v|, |u_x|, |v_x| \leq \max_{x \in I(0)} \{|\varphi_i|, |\partial_x \varphi_i|\} \exp(5Ht). \quad (3.17)$$

If we choose sufficiently small initial data, the above lemma tells us that the lifespan of the smooth solutions could be large and the solutions still keep small in the region of local existence. Based on this property, we begin to derive the *a priori* estimates in the region $t \geq t_0$ with large t_0 to be determined. For each $P(x^*, t^*)$, $t^* > t_0$, we denote

$$\Delta_P = \{(x, t) : \Gamma_2(t; x^*, t^*) \leq x \leq \Gamma_1(t; x^*, t^*), t_0 \leq t \leq t^*\},$$

and $I(t_0) = [\Gamma_2(t_0), \Gamma_1(t_0)]$, where $\Gamma_i(t) = \Gamma_i(t; x^*, t^*)$, $i = 1, 2$, are the characteristic curves corresponding to λ_i , $i = 1, 2$, respectively, passing through the point $P(x^*, t^*)$.

Also we can assume that the system (3.6) has a C^1 smooth solutions with $r - s > 0$ satisfying the *a priori* assumptions:

$$-10a_0\varepsilon \leq s(x, t) < r(x, t) \leq 10a_0\varepsilon, \quad |\partial_x r(x, t)| \leq 10a_0\mu, \quad |\partial_x s(x, t)| \leq 10a_0\mu,$$

where ε, μ are small constants to be determined later and

$$a_0 = 2 + \sup_x \int_{t_0}^{\infty} k(x, s) ds < +\infty.$$

From the fomulas of \tilde{r}, \tilde{s} , it holds that

$$|\tilde{r}(x, t)| \leq 200a_0^2\mu\varepsilon, \quad |\tilde{s}(x, t)| \leq 200a_0^2\mu\varepsilon. \quad (3.18)$$

We now prove the following key lemma:

Lemma 3.3. *Assume that the assumptions (H1)-(H3) of Theorem 3.1 hold, and for any $x \in I(t_0)$,*

$$\begin{aligned} |r(x, t_0)| &\leq \varepsilon, & |s(x, t_0)| &\leq \varepsilon, \\ |\partial_x r(x, t_0)| &\leq \mu, & |\partial_x s(x, t_0)| &\leq \mu. \end{aligned}$$

Then, in Δ_P for $t > t_0$ and t_0 large enough,

$$\begin{aligned} |r(x, t)| &\leq a_0\varepsilon, & |s(x, t)| &\leq a_0\varepsilon, \\ |\tilde{r}(x, t)| &\leq a_0\mu\varepsilon, & |\tilde{s}(x, t)| &\leq a_0\mu\varepsilon. \end{aligned}$$

Proof. We divide the proof into two steps: estimating r, s in Step 1 and \tilde{r}, \tilde{s} in Step 2.

Step 1. We first estimate $r + s$ and $r - s$. For $r + s$, we introduce a control function for the system (3.11). Let

$$\phi_1 = 2\varepsilon \frac{k(x, t_0)t_0^2}{k(x, t)t^2} + \frac{\varepsilon}{k(x, t)t^2} \int_{t_0}^t k^2(x, s)s^2 ds,$$

then

$$\begin{aligned}\phi_1 &\leq \frac{2\varepsilon k(x, t_0)t_0^{1+\delta}t_0^{1-\delta}}{k(x, t)t^{1+\delta}t^{1-\delta}} + \varepsilon \frac{kt^2kt^{2\delta}}{kt^2} \int_{t_0}^t s^{-2\delta} ds \\ &\leq \frac{C\varepsilon}{t^\delta} + \frac{\varepsilon kt}{1-2\delta} \leq C\varepsilon kt \leq C\varepsilon,\end{aligned}$$

due to the fact that $kt^{1+\delta}, \delta \in (0, \frac{1}{2})$ is increasing with respect to t for any fixed x . It is straightforward to check that ϕ_1 satisfies

$$\begin{aligned}\partial_t \phi_1 &= -\phi_1 \left(\frac{k_t}{k} + \frac{2}{t} \right) + \varepsilon k, \\ \partial_x \phi_1 &= -\frac{k_x}{k} \phi_1 + 2\varepsilon \frac{\partial_x k(x, t_0)}{k(x, t_0)} \frac{k(x, t_0)t_0^2}{k(x, t)t^2} + \frac{\varepsilon}{k(x, t)t^2} \int_{t_0}^t 2k(x, s) \partial_x k(x, s) s^2 ds,\end{aligned}\tag{3.19}$$

which, together with (H2), yields that $|\partial_x \phi_1| \leq C\varepsilon$. From (3.11)₁ and (3.19), we have

$$\partial_\beta(r + s - \phi_1) = (r + s - \phi_1) \left[-\left(\frac{2B_t}{B} + \frac{k_t}{k} \right) - BB_t k^2 r s \right] + R_1,$$

where

$$R_1 = -\partial_\beta \phi_1 - \phi_1 \left[\left(\frac{2B_t}{B} + \frac{k_t}{k} \right) + BB_t k^2 r s \right] - k\tilde{s} - 2 \left(\frac{B_x}{B} + \frac{3k_x}{2k} \right) krs + \frac{k_x}{2k} k(r^2 + s^2).$$

Again using (3.19) and Lemma 3.1, we obtain for small ε and μ that

$$R_1 \leq -\varepsilon k(1 - C\varepsilon - C\mu) < 0.\tag{3.20}$$

It should be noted that the *a priori* assumption (3.18) for the derivative \tilde{s} plays key role in the analysis of (3.20). Then we get

$$\partial_\beta(r + s - \phi_1) \leq (r + s - \phi_1) \left[-\left(\frac{2B_t}{B} + \frac{k_t}{k} \right) - BB_t k^2 r s \right]$$

which yields $r + s \leq \phi_1$ since $\phi_1(t_0) = 2\varepsilon \geq |r + s|(x, t_0)$. Similarly, we can prove $r + s \geq -\phi_1$. Thus $|r + s| \leq \phi_1$ holds. Note that ϕ_1 also satisfies

$$\phi_1 \leq 2\varepsilon + \varepsilon \int_{t_0}^t k ds \leq a_0 \varepsilon.\tag{3.21}$$

It then holds that

$$|r + s| \leq \min\{a_0 \varepsilon, C\varepsilon kt\}.\tag{3.22}$$

For $r - s$, let

$$\phi_2 = \varepsilon \left(2 + \int_{t_0}^t k(x, s) ds \right),$$

then $|\phi_2| \leq a_0 \varepsilon$. From (3.8)₁, we get

$$\partial_\beta(r - s - \phi_2) = Q(r - s - \phi_2) + R_2,$$

where

$$R_2 = -\partial_\beta \phi_2 + \phi_2 Q + k\tilde{s}.$$

Since $\partial_t \phi_2 = \varepsilon k$ and

$$|\partial_x \phi_2| = \varepsilon \left| \int_{t_0}^t k_x ds \right| \leq C\varepsilon \int_{t_0}^t k ds \leq C\varepsilon,$$

the formula (3.10) for Q implies $|Q| \leq C\varepsilon k$. It then holds that

$$R_2 \leq -\varepsilon k(1 - C\mu - C\varepsilon) < 0,$$

by choosing ε and μ small enough. Therefore

$$\partial_\beta(r - s - \phi_2) < Q(r - s - \phi_2),$$

which leads to $r - s \leq \phi_2$ due to $\phi_2(t_0) = 2\varepsilon \geq |r - s|(x, t_0)$. Similarly, $r - s \geq -\phi_2$ holds. Thus,

$$|r - s| \leq \phi_2 \leq a_0\varepsilon.$$

Therefore, we have

$$|r| \leq \frac{|r + s|}{2} + \frac{|r - s|}{2} \leq a_0\varepsilon, \quad |s| \leq \frac{|r + s|}{2} + \frac{|r - s|}{2} \leq a_0\varepsilon.$$

Step 2. Before deriving the estimates on \tilde{r} and \tilde{s} , we introduce a useful comparison principle for hyperbolic system, which is a variant of Lemma 2.2 of [24].

Lemma 3.4 (Hong [24]). *Let (u, v) be the C^1 smooth solutions to (3.15) in Δ_P with the following conditions:*

$$\begin{aligned} a_{12}(x, t) &\leq 0, \quad a_{21}(x, t) \leq 0 \text{ in } \Delta_P, \\ \varphi_1(x) &\leq 0 (\geq 0), \quad \varphi_2(x) \geq 0 (\leq 0) \text{ for all } x \in [\Gamma_2(t_0; x^*, t^*), \Gamma_1(t_0; x^*, t^*)], \\ R_1(x, t) &< 0 (> 0), \quad R_2(x, t) > 0 (< 0) \text{ in } \Delta_P. \end{aligned}$$

Then for $(x, t) \in \Delta_P$ with $t > t_0$, $u(x, t) < 0 (> 0)$, $v(x, t) > 0 (< 0)$.

Now we define another control function

$$\phi_3 = \mu\varepsilon \left(2 + \int_{t_0}^t k(x, s) ds \right),$$

which satisfies $\phi_3 \leq a_0\mu\varepsilon$. Then we can derive the following equations for $\tilde{r} - \phi_3$ and $\tilde{s} + \phi_3$:

$$\begin{aligned} \partial_\beta(\tilde{r} - \phi_3) &= (Q + f_r - k_x s)(\tilde{r} - \phi_3) + f_s(\tilde{s} + \phi_3) + R_3, \\ \partial_\alpha(\tilde{s} + \phi_3) &= g_r(\tilde{r} - \phi_3) + (Q + g_s - k_x r)(\tilde{s} + \phi_3) + R_4, \end{aligned}$$

where

$$\begin{aligned} R_3 &= -\partial_\beta \phi_3 + \phi_3(Q + f_r - k_x s - f_s) + (r - s)\delta_x f, \\ R_4 &= \partial_\alpha \phi_3 + \phi_3(g_r - Q - g_s + k_x r) + (r - s)\delta_x g. \end{aligned}$$

From (3.14) and the estimate $|r + s| \leq C\varepsilon kt$, one has

$$|\delta_x f| \leq \frac{C}{t}|r + s| + C\varepsilon^2 k \leq C\varepsilon k.$$

Similarly, we have $|\delta_x g| \leq C\varepsilon k$. A direct calculation yields that $\partial_t \phi_3 = \mu\varepsilon k$ and

$$|\partial_x \phi_3| = \mu\varepsilon \int_{t_0}^t |k_x| ds \leq C\mu\varepsilon \int_{t_0}^t k ds \leq C\mu\varepsilon.$$

From (3.13), $f_r - f_s$ is bounded by $C\varepsilon k$. Since $|Q| \leq C\varepsilon k$ and $|k_x s| \leq C\varepsilon k$, then

$$R_3 \leq (-\mu + C\varepsilon + C\mu\varepsilon)\varepsilon k < 0,$$

$$R_4 \geq (\mu - C\varepsilon - C\mu\varepsilon)\varepsilon k > 0,$$

if ε is smaller than μ .

Noting that $\partial_t \ln(kt^{1+\delta}) \geq 0$, one has

$$f_s = -\left(\frac{B_t}{B} + \frac{k_t}{2k}\right) + h.o.t \leq -\frac{1}{t} + o\left(\frac{1}{t}\right) + \frac{1+\delta}{2t} = -\frac{1-\delta}{2t} + o\left(\frac{1}{t}\right) \leq 0.$$

Similarly we can prove $g_r \leq 0$. The initial data satisfies the following estimates:

$$|\tilde{r}(x, t_0)| \leq 2\mu\varepsilon = \phi_3(t_0) \text{ and } |\tilde{s}(x, t_0)| \leq 2\mu\varepsilon = \phi_3(t_0).$$

Then, from Lemma 3.4, we have $\tilde{r} \leq \phi_3$ and $\tilde{s} \geq -\phi_3$. Similarly we can deduce that $\tilde{r} \geq -\phi_3$, and $\tilde{s} \leq \phi_3$ by considering the equations for $\tilde{r} + \phi_3$ and $\tilde{s} - \phi_3$. Therefore we have

$$|\tilde{r}| \leq \phi_3 \leq a_0\mu\varepsilon \text{ and } |\tilde{s}| \leq \phi_3 \leq a_0\mu\varepsilon.$$

□

Remark 3.2. The estimate

$$|r + s| \leq C\varepsilon kt$$

implies that $r + s$ decays to zero due to $k = o(\frac{1}{|t|})$, while $r - s$ may not decay. This means that $w + z$ decays faster than $w - z$ since $w = kr, z = rs$. Thus, we have verified this property observed in the ODE system (2.5) even for the general Gauss-Codazzi system (3.6). This property plays an essential role to close the *a priori* assumptions of derivatives \tilde{r}, \tilde{s} as shown above.

Next we estimate the lower bound of $r - s$ and upper bound of r_x, s_x .

Lemma 3.5. *Under the conditions of Lemma 3.3, and if for any $x \in I(t_0)$,*

$$(r - s)(x, t_0) \geq \varepsilon,$$

then in Δ_P for $t > t_0$ with t_0 sufficiently large,

$$(r - s)(x, t) \geq \frac{1}{2}\varepsilon,$$

$$|\partial_x r(x, t)| \leq 2a_0\mu, \quad |\partial_x s(x, t)| \leq 2a_0\mu,$$

$$|\partial_t r(x, t)| \leq C\varepsilon(k + t|k_t|), \quad |\partial_t s(x, t)| \leq C\varepsilon(k + t|k_t|).$$

Proof. Let

$$\phi_4 = \varepsilon \left(1 - 3a_0\mu \int_{t_0}^t k(x, s) ds \right),$$

then for small μ ,

$$\varepsilon \geq \phi_4 \geq \frac{1}{2}\varepsilon.$$

From (3.8), we get

$$\partial_\beta(r - s - \phi_4) = Q(r - s - \phi_4) + R_5,$$

where

$$R_5 = -\partial_\beta\phi_4 + Q\phi_4 + k\tilde{s}.$$

Again we know that $\partial_t\phi_4 = -3a_0\mu\varepsilon k$ and

$$|\partial_x\phi_4| = 3a_0\mu\varepsilon \left| \int_{t_0}^t k_x ds \right| \leq C\mu\varepsilon.$$

Thus, from Lemma 3.3,

$$\begin{aligned} R_5 &\geq 3a_0\mu\varepsilon k - C\mu\varepsilon^2 k - a_0\mu\varepsilon k - C\varepsilon^2 k \\ &= \varepsilon k(2a_0\mu - C\varepsilon\mu - C\varepsilon) > 0, \end{aligned}$$

by choosing ε sufficiently smaller than μ . Therefore,

$$\partial_\beta(r - s - \phi_4) > Q(r - s - \phi_4),$$

which gives

$$r - s \geq \phi_4 \geq \frac{1}{2}\varepsilon.$$

Finally, we obtain

$$\begin{aligned} |\partial_x r(x, t)| &\leq \frac{|\tilde{r}|}{r - s} \leq \frac{2a_0\mu\varepsilon}{\varepsilon} = 2a_0\mu, \\ |\partial_x s(x, t)| &\leq \frac{|\tilde{s}|}{r - s} \leq \frac{2a_0\mu\varepsilon}{\varepsilon} = 2a_0\mu. \end{aligned}$$

Note that

$$r_t = f - ksr_x \text{ and } s_t = g - krs_x,$$

then it is straightforward to deduce that

$$\begin{aligned} |r_t| &\leq \left| \frac{B_t}{B} + \frac{k_t}{2k} \right| |r + s| + C\varepsilon k \leq C\varepsilon(k + t|k_t|), \\ |s_t| &\leq \left| \frac{B_t}{B} + \frac{k_t}{2k} \right| |r + s| + C\varepsilon k \leq C\varepsilon(k + t|k_t|). \end{aligned}$$

This completes the proof. □

We now prove the Theorem 3.1.

Proof of Theorem 3.1. First, we choose sufficiently large $T = t_0$ such that Lemma 3.3 and Lemma 3.5 hold. Consider the Cauchy problem for system (3.6) with initial data $r(x, 0) = \eta_0, s(x, 0) = -\eta_0$. Denote

$$H = 1 + \sup_{\substack{|t| < T \\ x \in \mathbb{R}}} \{ |\partial_x^i \partial_t \ln B|, |\partial_x^i \partial_t \ln k|, |\partial_x^i (k \partial_x \ln B)|, |\partial_x^i k|, |\partial_x^2 k|, |\partial_x^i (BB_t k^2)|, i = 0, 1 \}.$$

Let η_0 be a small constant such that

$$\eta_0 \exp(30HT) \leq \varepsilon < 1.$$

From Lemma 3.2, one has

$$|r(x, t)| \leq \varepsilon, |s(x, t)| \leq \varepsilon; \quad |\partial_x r(x, t)| \leq \varepsilon \leq \sqrt{\varepsilon} =: \mu, |\partial_x s(x, t)| \leq \varepsilon \leq \sqrt{\varepsilon} =: \mu$$

for all x in \mathbb{R} , $0 < t \leq T$. On the other hand, integrating (3.8) together with the above estimates yields the following:

$$(r - s)(x, T) \geq \frac{\eta_0}{C(T)} \geq \frac{\varepsilon}{C(T)}$$

for some constant $C(T)$. Then by Lemmas 3.3 and 3.5, there exists a unique solution to the system (3.6) with initial data $r(x, T), s(x, T)$ in the region $t > T$ satisfying

$$\begin{aligned} -a_0\varepsilon &\leq s(x, t) < r(x, t) \leq a_0\varepsilon, \\ |(r - s)\partial_x r(x, t)| &\leq a_0\mu\varepsilon, \quad |(r - s)\partial_x s(x, t)| \leq a_0\mu\varepsilon, \\ |\partial_t r(x, t)| &\leq C\varepsilon(k + t|k_t|), \quad |\partial_t s(x, t)| \leq C\varepsilon(k + t|k_t|), \\ (r - s)(x, t) &\geq \frac{\varepsilon}{2C(T)}, \end{aligned}$$

for $(x, t) \in \mathbb{R} \times [T, \infty)$. Thus we can extend the solutions to the whole upper plane and get that $r(x, t), s(x, t) \in C^1(\mathbb{R} \times [0, \infty))$ with $r - s > 0$ everywhere. For the lower half plane, we can also get $r(x, t), s(x, t) \in C^1(\mathbb{R} \times (-\infty, 0])$ in the same way. Therefore, the existence of the system (3.6) in the whole plane with initial data $r(x, 0) = \eta_0, s(x, 0) = -\eta_0$ is proved. The argument in Section 2 and the relation $w = kr, z = ks$ yield the global existence of C^1 solutions to the system (2.1). Furthermore, the fundamental theorem of surface theory implies that there exists a C^3 surface with the prescribed metric whose Gauss curvature satisfying (H1)-(H3). This completes the proof of Theorem 3.1. \square

4. PROOF OF THEOREM 1.1 IN GEODESIC POLAR COORDINATES

In this section, we shall apply the Theorem 3.1 on the existence of isometric immersion in geodesic coordinates (x, t) to prove the main Theorem 1.1 in geodesic polar coordinates (θ, ρ) . The idea of the proof follows Hong [24] (cf. [19]) closely with the modifications that are necessary due to the slower decay rate at infinity of the Gauss curvature. For the sake of completeness we shall provide an outline of the proof.

The metric under the geodesic polar coordinates (θ, ρ) is of the form

$$\mathfrak{g} = G^2(\theta, \rho)d\theta^2 + d\rho^2$$

satisfying

$$\begin{cases} G_{\rho\rho} = k^2 G, \\ G(\theta, 0) = 0, G_\rho(\theta, 0) = 1, \text{ for any } \theta \in [0, 2\pi]. \end{cases} \quad (4.1)$$

Then the system (2.3) in the polar coordinates (θ, ρ) becomes

$$\begin{aligned} \bar{w}_\rho + \bar{z}\bar{w}_\theta &= -\left(\frac{G_\rho}{G} - \frac{k_\rho}{2k}\right)\bar{w} - \left(\frac{G_\rho}{G} + \frac{k_\rho}{2k}\right)\bar{z} - GG_\rho\bar{w}^2\bar{z} - \left(\frac{G_\theta}{G} + \frac{k_\theta}{2k}\right)\bar{w}\bar{z} + \frac{k_\theta}{2k}\bar{w}^2, \\ \bar{z}_\rho + \bar{w}\bar{z}_\theta &= -\left(\frac{G_\rho}{G} - \frac{k_\rho}{2k}\right)\bar{z} - \left(\frac{G_\rho}{G} + \frac{k_\rho}{2k}\right)\bar{w} - GG_\rho\bar{w}\bar{z}^2 - \left(\frac{G_\theta}{G} + \frac{k_\theta}{2k}\right)\bar{w}\bar{z} + \frac{k_\theta}{2k}\bar{z}^2, \end{aligned} \quad (4.2)$$

which is equivalent to the system (2.1) for smooth solutions if $\bar{w} > \bar{z}$, where we use $\bar{w}(\theta, \rho), \bar{z}(\theta, \rho)$ to denote $w(x, t), z(x, t)$ in the geodesic polar coordinates (θ, ρ) .

First we have the following properties of G in the polar coordinates similar to Lemma 3.1 and the proof is omitted.

Lemma 4.1. *Assume that the assumptions (A1)-(A2) hold, then*

$$\begin{aligned} \frac{G_\rho}{G} &= \frac{1}{\rho} + o\left(\frac{1}{\rho}\right), \\ \partial_\theta^i \ln G, i = 1, 2, \quad \rho \partial_\rho \partial_\theta \ln G &\text{ are bounded for sufficiently large } \rho, \\ 1 \leq \partial_\rho G \leq b_0, \quad \rho \leq G \leq b_0 \rho, \quad \rho \partial_\rho \ln G &> 1, \end{aligned}$$

where $b_0 = \exp \left\{ \sup_\theta \int_0^\infty \rho k^2 d\rho \right\}$.

As in Section 3, we also make the following variable transformations:

$$\bar{r} = \frac{\bar{w}}{k}, \quad \bar{s} = \frac{\bar{z}}{k},$$

then (4.2) can be rewritten as

$$\begin{aligned} \bar{r}_\rho + k\bar{s}\bar{r}_\theta &= -\left(\frac{G_\rho}{G} + \frac{k_\rho}{2k}\right)(\bar{r} + \bar{s}) - \left(\frac{G_\theta}{G} + \frac{3k_\theta}{2k}\right)k\bar{r}\bar{s} + \frac{k_\theta}{2k}k\bar{r}^2 - GG_\rho k^2 \bar{r}^2 \bar{s} \\ &=: f(\bar{r}, \bar{s}, \theta, \rho), \\ \bar{s}_\rho + k\bar{r}\bar{s}_\theta &= -\left(\frac{G_\rho}{G} + \frac{k_\rho}{2k}\right)(\bar{r} + \bar{s}) - \left(\frac{G_\theta}{G} + \frac{3k_\theta}{2k}\right)k\bar{r}\bar{s} + \frac{k_\theta}{2k}k\bar{s}^2 - GG_\rho k^2 \bar{s}^2 \bar{r} \\ &=: g(\bar{r}, \bar{s}, \theta, \rho). \end{aligned} \quad (4.3)$$

The system (4.3) has the same form as the system (3.6) for which the Theorem 3.1 holds. However, Theorem 3.1 can not be directly applied to the system (4.3) due to the following two reasons: (i) The condition (H3) fails to be true, and (ii) The coefficients in the system (4.3) are singular at the center $\rho = 0$.

To prove the main Theorem 1.1, we first consider the Gauss-Codazzi system in a neighbourhood of the center $\rho = 0$, denoted by Ω_1 in the geodesic coordinates, and then solve the Gauss-Codazzi system in Ω_2 that is the complement of Ω_1 in \mathbb{R}^2 in the geodesic polar coordinates.

As in previous section, we only consider the part $t \geq 0$. We construct the domains Ω_1 and Ω_2 as follows. Set

$$t_0(x) = R(1 + x^2), \quad \Omega_1 = \{(x, t) | 0 \leq t \leq t_0(x)\}, \quad \Omega_2 = \mathbb{R}_+^2 \setminus \Omega_1,$$

where R is a constant to be determined. Then we have

$$\partial\Omega_2 = \{(x, t) | t = t_0(x)\}.$$

Note that Ω_1 constructed above is different from that of [19] or [24].

Similar to Lemma 3.3 and Lemma 3.5, we can prove the existence of global smooth solution to any generalized Cauchy problem in the region $\{\rho > R\}$ for (4.3). Precisely, suppose that Ω is an unbounded domain with a smooth boundary curve:

$$\Omega = \{(\theta, \rho); \theta_1(\rho) < \theta < \theta_2(\rho), \rho > R\} \subset \{(\theta, \rho) : \rho > R, 0 < \theta < \pi\},$$

for some smooth functions $\theta_1 < \theta_2$ in $[R, +\infty)$. We have the following lemma.

Lemma 4.2. *Assume that $\partial\Omega$ is space-like with respect to ρ , i.e.,*

$$\partial_\rho \theta_1(\rho) < k(\theta_1(\rho), \rho) \bar{s}(\theta_1(\rho), \rho), \quad \partial_\rho \theta_2(\rho) > k(\theta_2(\rho), \rho) \bar{r}(\theta_2(\rho), \rho),$$

and on $\partial\Omega$,

$$-\varepsilon \leq \bar{s} < \bar{r} \leq \varepsilon, \quad \bar{r} - \bar{s} \geq \varepsilon, \quad |\partial_\theta \bar{r}|, |\partial_\theta \bar{s}| \leq \mu. \quad (4.4)$$

Then there are two constants R_0 and ε_0 such that when $R \geq R_0$ and $0 < \varepsilon < \varepsilon_0$, there exists a global smooth solution (\bar{r}, \bar{s}) to the problem (4.3)-(4.4) in the closure of Ω .

Lemma 4.2 can be applied for Ω_2 , provided that the conditions of Lemma 4.2 are satisfied. On the other hand, as in Lemma 3.2, we can find a small smooth solution in Ω_1 as long as the initial data are sufficiently small. To match the solutions in Ω_1 and Ω_2 , we use a coordinate transformation F from the geodesic coordinates (x, t) to the geodesic polar coordinates (θ, ρ) . As in [19] or [24], we have the following conclusions, the proofs of which are omitted:

- (1) $t, \frac{1}{2}|x| \leq \rho(x, t) \leq t + |x|$, and then $t + |x| \leq 3\rho$.
- (2) $\rho_t = \tanh \Phi$, $\theta_t = \frac{\xi}{G \cosh \Phi}$, $\rho_x = -\frac{\xi B}{\cosh \Phi}$, $\theta_x = \frac{B}{G} \tanh \Phi$ with $\Phi = \int_0^t \partial_\rho \ln G ds$ for $x \neq 0$.
- (3) The change of functions $(r, s) \mapsto (\bar{r}, \bar{s})$ satisfies

$$(\rho_t + k r \rho_x) k \bar{r} = \theta_t + k r \theta_x, \quad (\rho_t + k s \rho_x) k \bar{s} = \theta_t + k s \theta_x,$$

and it makes sense in a neighbourhood of $\partial\Omega_2$ in Ω_1 , provided that R is large enough.

(4) For any normalized vector V , define the differential map

$$F_*(V) = (\rho_t + \zeta \rho_x)(\partial_\rho + \bar{\zeta} \partial_\theta)$$

with

$$\bar{\zeta} = \frac{\theta_t + \zeta \theta_x}{\rho_t + \zeta \rho_x} \text{ if } \rho_t + \zeta \rho_x \neq 0.$$

And denote $\bar{F}_*(\zeta) = \bar{\zeta}$ if it makes sense. Then it holds that

$$k\bar{r} = \bar{F}_*(kr), \quad k\bar{s} = \bar{F}_*(ks), \quad (4.5)$$

and

$$\bar{r} - \bar{s} = \frac{B(r - s)}{G(\rho_t + kr\rho_x)(\rho_t + ks\rho_x)}. \quad (4.6)$$

To seek the initial data, we choose two smooth even functions h_1 and h_2 in \mathbb{R} satisfying

$$h_1(\zeta) \geq 1 + \sup_{\substack{(x,t) \in \Omega_1 \\ |x| \leq \zeta \\ i=0,1}} \{ |\partial_x^i \partial_t \ln B|, |\partial_x^i \partial_t \ln k|, |\partial_x^i (k \partial_x \ln B)|, |\partial_x^2 k|, |\partial_x^i k|, |\partial_x^i (BB_t k^2)| \},$$

and

$$h_2(\zeta) \geq 1 + h_1(\zeta) + \sup_{(x,t) \in \Omega_1, |x| \leq \zeta} \left\{ \frac{|t'_0(x)|}{t_0(x) + R} \right\},$$

and increasing in \mathbb{R}_+ . Then we define $\eta(x)$ for $x > 0$ by

$$\eta(x) = \frac{1}{8\pi} \int_x^\infty \psi(\zeta) \frac{\exp\{-30(t_0(\zeta + 1) + R)h_1(\zeta + 1) - \zeta^2\}}{(t_0(\zeta + 1) + R)h_2(\zeta + 1)} d\zeta,$$

where ψ is a smooth cutoff function such that $\psi(\zeta) = 1$ for $\zeta \geq 1$ and $\psi(\zeta) = 0$ for ζ near zero. And we can extend η as a smooth even function in \mathbb{R} . Finally, the initial data is chosen as

$$r(x, 0) = \sigma \eta(x), \quad s(x, 0) = -\sigma \eta(x), \quad (4.7)$$

with small positive constant σ to be determined later. Thus, for the Cauchy problem for (3.6) with (4.7) in Ω_1 , we obtain the following lemma.

Lemma 4.3. *There exists a positive $\sigma_0 \in (0, 1)$ such that the Cauchy problem of (3.6) with (4.7) in Ω_1 admits a smooth solution (r, s) with $r > s$ for any $\sigma \in (0, \sigma_0)$. Furthermore, on $\partial\Omega_2$, the solution satisfies (4.4).*

Once we have shown Lemma 4.3, we can complete the proof of Theorem 1.1 as in [19] or [24]. In fact, assume (r, s) is the solution of (3.6) with (4.7) as in Lemma 4.3. Since the change of functions $(r, s) \mapsto (\bar{r}, \bar{s})$ makes sense in a neighbourhood of $\partial\Omega_2$ in Ω_1 , and Lemma 4.3 guarantees that the conditions in Lemma 4.2 are satisfied for $\Omega = F(\Omega_2)$, there is a smooth solution (\bar{r}, \bar{s}) to the problem (4.3)-(4.4). Taking the variable transformation:

$$(\theta, \rho) \in F(\Omega_2) \mapsto (x, t) \in \Omega_2 \text{ and } (\bar{r}, \bar{s}) \mapsto (r, s),$$

again from Lemma 4.2, we have a smooth extension of (r, s) in Ω_1 to Ω_2 , with $r > s$ everywhere. Therefore, we obtain a global smooth solution of (3.6) with (4.7) with $r > s$ for $t > 0$, which yields a smooth isometric immersion of g into \mathbb{R}^3 by applying fundamental theorem of surface theory. Therefore it remains to verify Lemma 4.3, the proof of which is different from [19] or [24].

The proof of Lemma 4.3: Similarly to [19], it is easy to verify $r > s$ on $\partial\Omega_2$ and $\bar{r} > \bar{s}$ on $\partial F(\Omega_2)$. We can show that the differential system (3.6) with (4.7) in Ω_1 has a global smooth solution (r, s) for each x and each $\sigma \in (0, \sigma_0)$ with $\sigma_0 \leq 1$, satisfying

$$|r|, |s|, |r_x|, |s_x| \leq \frac{C\sigma}{(t_0(x) + R)h_2(x)} \quad (4.8)$$

in the characteristic triangle where the solution exists. Furthermore, we can also prove that $\partial\Omega_2$ is space-like in ρ . The proof is omitted.

To complete the proof of Lemma 4.3, it remains to show that (r, s) satisfies (4.4). From (4.5), we have

$$\bar{r} = \frac{\bar{F}_*(kr)}{k} = \frac{\xi + kBr \sinh \Phi}{kG(\sinh \Phi - kBr\xi)}, \quad \bar{s} = \frac{\bar{F}_*(ks)}{k} = \frac{\xi + kB s \sinh \Phi}{kG(\sinh \Phi - kB s \xi)}.$$

Since $0 \leq 1 - \rho_t \leq \frac{1}{\cosh \Phi}$, then, for $|x| > 2$,

$$\Phi = \int_0^t \frac{G_\rho}{G} ds \geq \int_0^t \frac{1}{\rho} ds \geq \int_0^t \frac{\rho_s}{\rho} ds = \ln \rho(x, t) - \ln \rho(x, 0),$$

and thus $e^\Phi \geq \frac{\rho(x, t)}{\rho(x, 0)}$. From (A1), we also have $k \geq \frac{C}{\rho^{1+\delta}}$, and then on $\partial\Omega_2 \cap \{|x| > 2\}$,

$$|\bar{r}| \leq \frac{1}{2kG \sinh \Phi} \leq \frac{C\rho(x, 0)\rho^{1+\delta}(x, t)}{\rho^2(x, t)} \leq \frac{C|x|}{t_0(x)^{1-\delta}} \leq \frac{C|x|}{R^{1-\delta}|x|^{2-2\delta}} \leq \frac{C}{R^{1-\delta}} \leq \varepsilon,$$

due to $0 < \delta < 1/2$. Meanwhile, on $\partial\Omega_2 \cap \{|x| \leq 2\}$, one has

$$|\bar{r}| \leq \frac{1}{2kG \sinh \Phi} \leq \frac{C\rho^\delta}{R} \leq \frac{C + CR^\delta}{R} \leq \varepsilon.$$

Therefore, on $\partial\Omega_2$, $|\bar{r}| \leq \varepsilon$. Similarly, $|\bar{s}| \leq \varepsilon$.

We also need to estimate $\partial_\theta \bar{r}$ and $\partial_\theta \bar{s}$. Taking the partial derivative of \bar{r} with respect to θ , we get on $\partial\Omega_2$,

$$\begin{aligned} \partial_\theta \bar{r} = & - \left(\frac{k_\theta}{k} + \frac{G_\theta}{G} \right) \frac{1}{kG} \frac{\xi + kBr \sinh \Phi}{\sinh \Phi - kBr\xi} \\ & + \partial_\theta r \left[\frac{B \sinh \Phi}{G(\sinh \Phi - kBr\xi)} + \frac{B\xi(\xi + kBr \sinh \Phi)}{G(\sinh \Phi - kBr\xi)^2} \right] \\ & + \left(\frac{B_x}{B} x_\theta + \frac{B_t}{B} t_\theta + \frac{k_\theta}{k} \right) \left[\frac{rB \sinh \Phi}{G(\sinh \Phi - kBr\xi)} - \frac{B\xi(\xi + kBr \sinh \Phi)}{G(\sinh \Phi - kBr\xi)^2} \right] \\ & + \partial_\theta \Phi \left[\frac{Br \cosh \Phi}{G(\sinh \Phi - kBr\xi)} - \frac{\cosh \Phi(\xi + kBr \sinh \Phi)}{kG(\sinh \Phi - kBr\xi)^2} \right] = \sum_{j=1}^4 I_j. \end{aligned}$$

For I_1 , as above one has $|I_1| \leq \frac{C}{kG \sinh \Phi} \leq \frac{C}{R^{1-\delta}} \leq \mu$, with R large enough. For I_2 , we have on $\partial F(\Omega_2)$,

$$\begin{aligned} |\partial_\theta r| &= |r_t t_\theta + r_x x_\theta| \leq |f(\theta, \rho) t_\theta| + k|s||r_x||t_\theta| + |r_x||x_\theta| \\ &\leq \frac{C\sigma}{(t_0(x) + R)h_2(x)} \left[\frac{G}{B} \tanh \Phi + \frac{(h_1(x) + k)G}{\cosh \Phi} \right] \leq C\sigma, \end{aligned}$$

since $G \leq b_0\rho$, $\rho \leq 2t$ on $\partial\Omega_2$. Thus we obtain $|I_2| \leq C|\partial_\theta r| \leq C\sigma \leq \mu$ from (4.8) and choosing small σ_0 . In the same way, we have $|I_3| \leq C|r| \leq \mu$. For I_4 , it holds that

$$|I_4| = \left| \partial_\theta \Phi \frac{-\xi \cosh \Phi (1 + k^2 B^2 r^2)}{kG(\sinh \Phi - kBr\xi)^2} \right| \leq \frac{C \cosh \Phi}{kG \sin^2 \Phi} |\partial_\theta \Phi|.$$

We then derive the estimates of $\partial_\theta \Phi$ as follows. For $|x| > 2$,

$$\begin{aligned} \partial_\theta \Phi &= \int_0^t (\partial_\rho \ln G)_x ds x_\theta + \partial_\rho \ln G t_\theta \\ &= \left[\int_0^t (k^2 - (\partial_\rho \ln G)^2) \frac{-\xi B}{\cosh \Phi} ds + \int_0^t \partial_{\theta\rho} \ln G \frac{B \tanh \Phi}{G} ds \right] \frac{G}{B} \tanh \Phi \\ &\quad + \partial_\rho \ln G \frac{\xi G}{\cosh \Phi}. \end{aligned}$$

Moreover, it holds that

$$\begin{aligned} \left| \int_0^t k^2 \frac{-\xi B}{\cosh \Phi} ds \right| &\leq C \int_0^t k^2 s ds \leq C, \\ \left| \int_0^t \partial_{\theta\rho} \ln G \frac{B \tanh \Phi}{G} ds \right| &\leq \int_0^t \frac{Ct}{\rho^3} \int_0^\rho k^2 \alpha^2 d\alpha ds \leq \int_0^t Ck^2 s ds \leq C, \end{aligned}$$

and

$$\left| \int_0^t \frac{G_\rho^2}{G^2} \frac{-\xi B}{\cosh \Phi} ds \right| \leq \int_0^t \frac{C\rho(x,0)s}{\rho(x,s)^3} ds \leq C|x| \int_0^t \frac{1}{(s + |x|)^2} ds \leq C,$$

where we have used the fact: $t + |x| \leq 3\rho$. Hence, for $|x| > 2$,

$$|\partial_\theta \Phi| \leq \frac{CG}{B} \tanh \Phi + \frac{C}{\cosh \Phi}.$$

Therefore, on $\partial\Omega_2 \cap \{|x| > 2\}$,

$$\begin{aligned} |I_4| &\leq \frac{C \cosh \Phi}{kG \sinh^2 \Phi} \left(\frac{CG}{B} \tanh \Phi + \frac{C}{\cosh \Phi} \right) \\ &\leq \frac{C}{kB \sinh \Phi} + \frac{C}{kG \sinh^2 \Phi} \leq \frac{Ct^\delta \rho(x, 0)}{\rho(x, t)} + \frac{C\rho^2(x, 0)}{\rho^{2-\delta}} \leq \frac{C}{R^{1-\delta}} \leq \mu, \end{aligned}$$

for large R and small σ_0 .

When $|x| \leq 2$, on $\partial\Omega_2$,

$$B_t(x, \infty) \geq \inf_{|x| \leq 2} \int_0^t k^2(x, s) ds > 0.$$

Then following the idea in the proof of Lemma 3.1, we have, on $\partial\Omega_2 \cap \{|x| \leq 2\}$,

$$\left| \frac{B_t}{B} - \frac{1}{t} \right| \leq \frac{C t^{-1} \int_0^t B k^2 s ds}{t B_t(x, \infty)} \leq C k^2 t,$$

and

$$\frac{G_\rho}{G} - \frac{1}{\rho} \leq \frac{C \rho^{-1} \int_0^\rho G k^2 \rho ds}{\rho G_\rho(x, \infty)} \leq C k^2 \rho.$$

Recalling the formula of $\partial_\theta \Phi$ in [24],

$$\begin{aligned} \partial_\theta \Phi &= \xi \frac{G_\rho}{\cosh \Phi} + \left[\frac{1}{\rho} - \frac{1}{t} + \left(\frac{G_\rho}{G} - \frac{1}{\rho} \right) \right. \\ &\quad \left. + (\tanh \Phi - 1) \frac{G_\rho}{G} - \left(\frac{B_t}{B} - \frac{1}{t} \right) \right] \xi G \sinh \Phi, \end{aligned}$$

we have on $\partial\Omega_2 \cap \{|x| \leq 2\}$,

$$|\partial_\theta \Phi| \leq \frac{C}{R} + C k^2 R^2 \sinh \Phi,$$

since $\rho \leq t_0(x) + |x| \leq CR$ and $\frac{1}{t} - \frac{1}{\rho} \leq C/R^2$. Therefore, on $\partial\Omega_2 \cap \{|x| \leq 2\}$,

$$|I_4| \leq \frac{C \cosh \Phi}{kG \sinh^2 \Phi} \left(\frac{C}{R} + C k^2 R^2 \sinh \Phi \right) \leq \frac{C}{kR^3} + CkR \leq \mu,$$

where we have used the fact: $\frac{C}{R^{1+\delta}} \leq k = o\left(\frac{1}{R}\right)$ on $\partial\Omega_2 \cap \{|x| \leq 2\}$, for R large enough. Finally, we get $|I_4| \leq \mu$ on $\partial\Omega_2$. Therefore, $|\partial_\theta \bar{r}| \leq \mu$ on $\partial\Omega_2$. Similarly, we have $|\partial_\theta \bar{s}| \leq \mu$, on $\partial\Omega_2$. The proof is complete. \square

ACKNOWLEDGMENTS

F. Huang's research was supported in part by NSFC Grant No. 11371349. D. Wang's research was supported in part by the NSF Grant DMS-1312800 and NSFC Grant No. 11328102. The authors would like to thank Professor Qin Han for his valuable discussions and suggestions.

REFERENCES

- [1] R. Bryant, P. Griffiths, D. Yang, Characteristics and existence of isometric embeddings, *Duke Math. J.* **50** (1983), 893-994.
- [2] Y. D. Burago and S. Z. Shefel, The geometry of surfaces in Euclidean spaces, Geometry III, 1-85, Encyclopaedia Math. Sci., 48, Burago and Zalgaller (Eds.), Springer-Verlag: Berlin, 1992.
- [3] W. Cao, F. Huang, D. Wang, Isometric Immersions of Surfaces with Two Classes of Metrics and Negative Gauss Curvature. *Arch. Ration. Mech. Anal.* **218** (2015), no. 3, 1431-1457.
- [4] W. Cao, F. Huang, D. Wang, Isometric immersion of surface with Gauss curvature and the Lax-Friedrichs scheme. To appear in *SIAM J. Math. Anal.*
- [5] E. Cartan, Sur la possibilité de plonger un espace Riemannien donné dans un espace Euclidien, *Ann. Soc. Pol. Math.* **6** (1927), 1-7.
- [6] G.-Q. Chen, J. Clelland, M. Slemrod, D. Wang, and D. Yang, Isometric embedding via strongly symmetric positive systems. arXiv:1502.04356 [math.DG], 2015.
- [7] G.-Q. Chen, M. Slemrod, D. Wang, Isomeric immersion and compensated compactness. *Commun. Math. Phys.* **294** (2010), 411-437.
- [8] C. Christoforou, BV weak solutions to Gauss-Codazzi system for isometric immersions. *J. Diff. Eqs.* **252** (2012), 2845-2863.
- [9] C. Christoforou, M. Slemrod, Isometric immersions via compensated compactness for slowly decaying negative Gauss curvature and rough data, *Z. Angew. Math. Phys.* **66** (2015), 3109-3122.
- [10] D. Codazzi, Sulle coordinate curvilinee duna superficie dello spazio, *Ann. Math. Pura Applata*, **2** (1860), 101-119.
- [11] G.-C. Dong, The semi-global isometric imbedding in \mathbb{R}^3 of two-dimensional Riemannian manifolds with Gaussian curvature changing sign cleanly. *J. Partial Differential Equations* **6** (1993), 62-79.
- [12] N.V. Efimov, Generation of singularities on surfaces of negative curvature. (Russian) *Mat. Sb. (N.S.)* **64** (106) (1964), 286-320.
- [13] N.V. Efimov, Surfaces with slowly varying negative curvature, *Russian Math. Survey* **21**(1966), 1-55.
- [14] J. B. Goodman and D. Yang, Local solvability of nonlinear partial differential equations of real principal type. Preprint, 1988.
- [15] M. Gromov, Partial Differential Relations, *Springer-Verlage, Berlin Heidelberg*, 1986.
- [16] M. Gromov, V.A. Rokhlin, Embeddings and immersions in Riemannian geometry, *Uspekhi Mat. Nauk.* **25**(1970), no. 5, 3-62; *Russian Math. Survey*, 25(1970), no. 5, 1-57.
- [17] P. Guan, Y. Li, The Weyl problem with nonnegative Gauss curvature, *J. Diff. Geometry* **39** (1994), 331-342.
- [18] Q. Han, On isometric embedding of surfaces with Gauss curvature changing sign cleanly. *Comm. Pure Appl. Math.* **58** (2005), 285-295.
- [19] Q. Han, J.-X. Hong, Isometric embedding of Riemannian manifolds in Euclidean spaces. Providence, RI: Amer. Math. Soc., 2006.

- [20] Q. Han, J.-X. Hong, C.-S. Lin, Local Isometric Embedding of Surfaces with Nonpositive Gaussian Curvature, *J. Differential Geom.* **63** (2003), 475-520.
- [21] P. Hartman, P. Winter, Gaussian curvature and local embedding, *Amer. J. Math.* **73** (1951), 876-884.
- [22] E. Heinz, On Weyl's embedding problem, *J. Math. Mech.* **11** (1962), 421-454.
- [23] D. Hilbert, Ueber flachen von constanter Gausscher Krümmung, *Trans. Amer. Math. Soc.* **2** (1901), 87-99.
- [24] J.-X. Hong, Realization in \mathbb{R}^3 of complete Riemannian manifolds with negative curvature. *Commun. Anal. Geom.* **1** (1993), 487-514.
- [25] J.-X. Hong, C. Zuily, Isometric embedding of the 2-sphere with nonnegative curvature in \mathbb{R}^3 , *Math. Z.* **219** (1995), 323-334.
- [26] M. Janet, Sur la possibilité de plonger un espace Riemannien donné dans un espace Euclidien. *Ann. Soc. Pol. Math.* **5** (1926), 38-43.
- [27] C.-S. Lin, The local isometric embedding in \mathbb{R}^3 of 2-dimensional Riemannian manifolds with nonnegative curvature, *J. Diff. Geometry* **21** (1985), 213-230.
- [28] C.-S. Lin, The local isometric embedding in \mathbb{R}^3 of 2-dimensional Riemannian manifolds with Gaussian curvature changing sign cleanly. *Comm. Pure Appl. Math.* **39** (1986), 867-887.
- [29] G. Mainardi, Su la teoria generale delle superficie. *Giornale dell' Istituto Lombardo* **9** (1856), 385-404.
- [30] G. Nakamura, Y. Maeda, Local isometric embedding problem of Riemannian 3-manifold into \mathbb{R}^6 . *Proc. Japan Acad. Ser. A Math. Sci.* **62** (1986), no. 7, 257-259.
- [31] G. Nakamura, Y. Maeda, Local smooth isometric embeddings of low-dimensional Riemannian manifolds into Euclidean spaces. *Trans. Amer. Math. Soc.* **313** (1989), no. 1, 1-51.
- [32] J. Nash, C^1 isometric imbeddings. *Ann. of Math.* (2) **60** (1954), 383-396.
- [33] J. Nash, The imbedding problem for Riemannian manifolds. *Ann. of Math.* (2) **63** (1956), 20-63.
- [34] L. Nirenberg, The Weyl and Minkowski problems in differential geometry in the large, *Comm. Pure Appl. Math.* **6** (1953), 337-394.
- [35] K. M. Peterson, Ueber die Biegung der Flächen, *Dorpat. Kandidatenschrift* 1853.
- [36] T. E. Poole, The local isometric embedding problem for 3-dimensional Riemannian manifolds with cleanly vanishing curvature, *Comm. in Partial Differential Equations* **35** (2010), 1802-1826.
- [37] È. G. Poznyak, E. V. Shikin, Small parameters in the theory of isometric imbeddings of two-dimensional Riemannian manifolds in Euclidean spaces. In: Some Questions of Differential Geometry in the Large, *Amer. Math. Soc. Transl. Ser. 2*, **176** (1996), 151-192, AMS: Providence, RI.
- [38] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer, New York, 1994.
- [39] H. Weyl, Über die Bestimmtheit einer geschlossenen konvex Fläche durch ihr Linienelement, *Vierteljahresschrift der nat.-Forsch. Ges. Zurich* **61** (1916), 40-72.
- [40] S.-T. Yau, Seminaire on differential geometry, *Princeton University Press*, 1982.

INSTITUTE OF APPLIED MATHEMATICS, AMSS, CAS, BEIJING 100190, CHINA.
E-mail address: cwt@amss.ac.cn

INSTITUTE OF APPLIED MATHEMATICS, AMSS, CAS, BEIJING 100190, CHINA.
E-mail address: fhuang@amt.ac.cn

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA 15260, USA.
E-mail address: dwang@math.pitt.edu